M E T U - N C C
Mathematics Group

	Calculus with Analytic Geometry Second Midterm Exam	
Code : MATH 119 Acad.Year : 2011 – 2012 Semester : Spring Coord. : S.D./H.T. Date : 28.04.2012 Time : 14.40 Duration : 120 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : 6 Questions on 6 Pages Total 100 Points	

Q1	Q2	Q3	Q4	Q5	Q6		

Q.1 ($12 + 3 = 15$ pts) By applying the Mean Value Theorem to $f(x) = \cos x + \frac{x^2}{2}$ on the interval $[0, x]$, and using the known inequality $x > \sin x$ for $x > 0$;

(a) Show that $\cos x > 1 - \frac{x^2}{2}$ for $x > 0$.

Let $f(x) = \cos x + \frac{x^2}{2}$. We have $f(0) = 1$ &

$f'(x) = -\sin x + x > 0$ for $x>0$ since $x > \sin x$ for $x>0$.

So $f(x)$ is increasing for all $x \in (0, \infty)$ & since $f(0)=1$ we get

$f(x) > 1$ for all $x>0$ $\iff$ $\cos x > 1 - \frac{x^2}{2}$ for all $x>0$

(b) Explain why the result of Part (a) is ALSO true for $x < 0$.

$f(x) = \cos x + \frac{x^2}{2}$ is an even function, so the result holds also for $x<0$

Q.2 ($4 \times 5 = 20$ pts) Consider the function $y = f(x) = \dfrac{x^2}{(x-2)^2}$.

(a) Write down the domain of the function, and find its asymptotes.

$\text{Dom} f = \mathbb{R} - \{2\}$

$\lim_{x \to \mp\infty} f(x) = 1$ $\qquad$ $\lim_{x \to 2^+} f(x) = \infty$

$\lim_{x \to 2^-} f(x) = \infty$

(b) Find intervals of increase and decrease.

$$f'(x) = \frac{2x(x-2)^2 - x^2 \cdot 2(x-2)}{(x-2)^4} = \frac{2x(x-2)(x-2-x)}{(x-2)^4} = \frac{-4x(x-}{(x-2)^4}$$

		0		2	
$f'(x)$	$-$	0	$+$		$-$
$f(x)$	Dec		Inc		Dec

(c) Find local maximum and minimum points if there is any.

$x = 0$ is a local min point

(d) Find intervals of concavity. Is there any inflection points?

$$f''(x) = \frac{-4(x-2)^3 + 4x \cdot 3(x-2)^2}{(x-2)^6} = \frac{8x+8}{(x-2)^4}$$

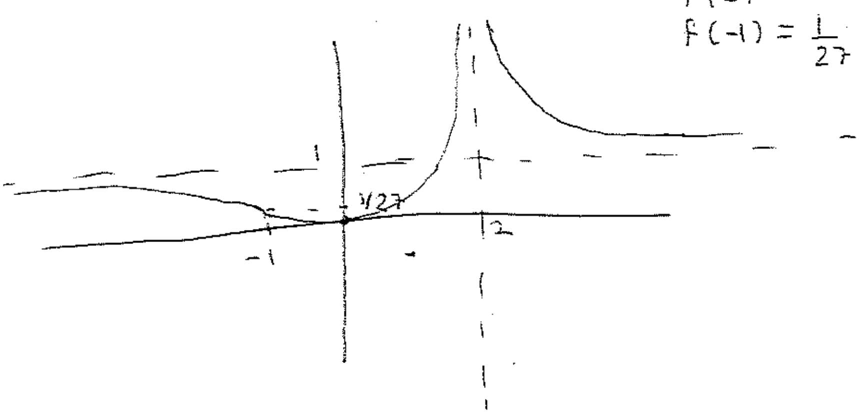
		-1		2	
$f''(x)$	$-$	0	$+$		$+$
$f(x)$	C.D.		C.U.		C.U.

Inf Point

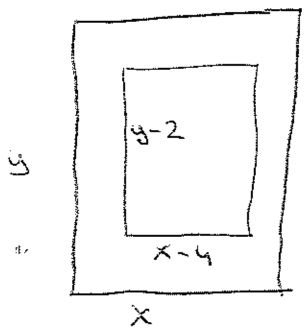
(e) Sketch its graph.

$f(0) = 0$

$f(-1) = \frac{1}{27}$



<u>Q.3</u> (15 pts) The top–bottom margins of a poster are $1cm$. and left–right margins are $2cm$. The printed area of the poster will be $32cm^2$. Find the dimensions x and y of the poster which minimize the total area A of the poster. (JUSTIFY your ANSWERS !!!)



$$(x-4)(y-2) = 32$$
$$xy - 4y - 2x + 8 = 32$$
$$y(x-4) = 2x + 24$$
$$y = \frac{2x+24}{x-4}$$

$$A = xy = \frac{2x^2+24x}{x-4} \quad , \quad x > 4.$$

$$A'(x) = \frac{(4x+24)(x-4) - (2x^2+24x)}{(x-4)^2}$$

$$A'(x) = \frac{2x^2 - 16x - 96}{(x-4)^2}$$

<u>Crit Points</u>

$x = 4$ & $2x^2 - 16x - 96 = 0$

$2(x^2 - 8x - 48) = 0$

$2(x-12)(x+4) = 0$

$\downarrow$ $x = 12$ $\quad$ $\downarrow$ $x = -4$

Since $x > 4$, the only crit. point is $x = 12$.

For $4 < x < 12$, $A'(x) < 0$

For $12 < x$, $A'(x) > 0$

$\Rightarrow$ So $A(x)$ has a global min at $x = 12$

Dimensions: $x = 12$ & $y = 6$.

Q.4 ($5 \times 3 = 15$ pts) This problem has three INDEPENDENT parts.

(a) Find $F(\pi/2)$ and $F'(x)$, if $F(x) = \int_{\sin x}^{1} \sqrt{1-t^2}\, dt$.

$$F(\pi/2) = \int_1^1 \sqrt{1-t^2}\, dt = 0$$

$$F(x) = -\int_1^{\sin x} \sqrt{1-t^2}\, dt \underset{\substack{\uparrow \\ \text{FTC}}}{\Rightarrow} F'(x) = -\sqrt{1-\sin^2 x} \cdot \cos x$$

(b) Find $F(0)$ and $F'(x)$, if $F(x) = \int_{x^2-1}^{\cos x} t^9 \sin(t^2)\, dt$.

$$F(0) = \int_{-1}^{1} t^9 \sin(t^2)\, dt = 0 \leftarrow \text{since } t^9\sin(t^2) \text{ is an odd function}$$

$$F(x) = -\int_0^{x^2-1} t^9 \sin(t^2)\, dt + \int_0^{\cos x} t^9 \sin(t^2)\, dt$$

$$F'(x) = -(x^2-1)^9 \sin((x^2-1)^2) \cdot 2x + \cos^9 x \cdot \sin(\cos^2 x) \cdot (-\sin x)$$

(c) An object moves along a line with a velocity $v(t) = -t^2 + 6t - 8$ at time t. Find the DISTANCE travelled by the object during the time period $0 \le t \le 4$.

$$\text{Distance Traveled} = \int_0^4 |v(t)|\, dt$$

$$v(t) = -t^2+6t-8 = -(t-2)(t-4)$$

	0		2		4
$v(t)$		$-$	ϕ	$+$	ϕ

So
$$\int_0^4 |v(t)|\, dt = -\int_0^2 (-t^2+6t-8)\, dt + \int_2^4 (-t^2+6t-8)\, dt$$

$$= \frac{1}{3}t^3 - 3t^2 + 8t \Big|_0^2 + -\frac{1}{3}t^3 + 3t^2 - 8t \Big|_2^4$$

$$= \left(\frac{8}{3} - 12 + 16\right) + \left(-\frac{64}{3} + 48 - 32\right) - \left(-\frac{8}{3} + 12 - 16\right)$$

Last Name:

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Q.5 ($7+8=15$ pts) Consider the plane region $\mathcal{R}$ lying to the RIGHT of the parabola $x-y^2+4=0$ and to the LEFT of the straightline $x-y+2=0$.
(a) Express the AREA of $\mathcal{R}$ as an integral in the x direction, *i.e.*, as an integral with respect to x. (DO NOT evaluate the integral)

$x = y^2 - 4$ & $x = y - 2$

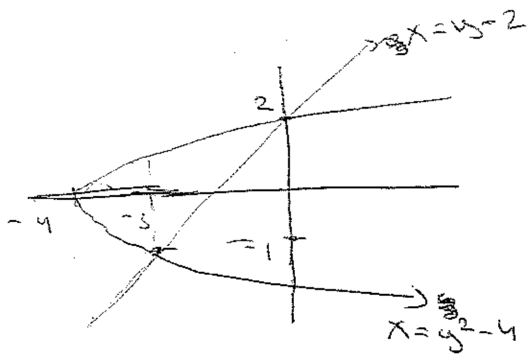
Intersection points

$y^2 - 4 = y - 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2$ & $y = -1$



$$A = \int_{-4}^{-3} \left(\sqrt{x+4} - (-\sqrt{x+4}) \right) dx + \int_{-3}^{0} \left(\sqrt{x+4} - (x+2) \right) dx$$

(b) Calculate the AREA of $\mathcal{R}$ by an integral in the y direction. (DO evaluate the integral finding its numerical value)

$$A = \int_{-1}^{2} \left[(y-2) - (y^2-4) \right] dy = \int_{-1}^{2} (-y^2 + y + 2)\, dy$$

$$= -\frac{y^3}{3} + \frac{y^2}{2} + 2y \Big|_{-1}^{2} = \left(-\frac{8}{3} + 2 + 4\right) - \left(+\frac{1}{3} + \frac{1}{2} - 2\right)$$

Q.6 ($5 \times 4 = 20$ pts) Evaluate the following integrals:

(a) $\int x^2(1+2x)^{1/3}\,dx = \int \frac{(u-1)^2}{8} u^{1/3}\,du = \frac{1}{8}\int \left(u^{7/3} - 2u^{4/3} + u^{1/3}\right) du$

$u = 1+2x \Rightarrow du = 2dx$

$x = \left(\frac{u-1}{2}\right)$

$= \frac{1}{8}\left(\frac{3}{10}(1+2x)^{10/3} - 2\cdot\frac{3}{7}(1+2x)^{7/3} + \frac{3}{4}(1\right.$

$+ C$

(b) $\int \frac{\cos(\sqrt{x}+3)}{\sqrt{x}}\,dx = 2\int \cos(u)\,du = 2\sin(\sqrt{x}+3) + C$

$u = \sqrt{x} + 3$

$du = \frac{1}{2\sqrt{x}}\cdot dx$

(c) $\int_{-\pi}^{\pi}\left[\sin(x^3)+1\right]dx = \int_{-\pi}^{\pi} \sin(x^3)\,dx + \int_{-\pi}^{\pi} 1\,dx = 0 + 2\pi$

$f(x) = \sin(x^3)$

$f(-x) = \sin(-x^3)$

$f(-x) = -\sin(x^3) = -f(x)$

odd function

(d) $\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{x^2+x^4}\,dx = \int_{-\sqrt{3}}^{\sqrt{3}} |x|\sqrt{1+x^2}\,dx = \int_{-\sqrt{3}}^{0} -x\sqrt{1+x^2}\,dx + \int_{0}^{\sqrt{3}} x\sqrt{1+x^2}\,dx$

$= -\frac{1}{3}(1+x^2)^{3/2}\Big|_{-\sqrt{3}}^{0} + \frac{1}{3}(1+x^2)^{3/2}\Big|_{0}^{\sqrt{3}}$

$\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int u^{1/2}\,du$

$u = 1+x^2$

$du = 2x\,dx$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{1}{3}(1+x^2)^{3/2} + C$