M E T U
Northern Cyprus Campus

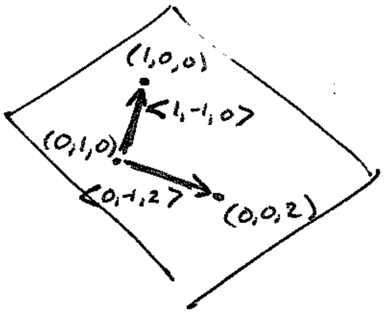
	Calculus for Functions of Several Variables II. Midterm	

Code : *Math 120* Acad.Year: *2009-2010* Semester : *Spring* Date : *8.5.2010* Time : *9:00* Duration : *120 minutes*	Last Name: Solutions Name : Student No Department: Section: Signature: 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8		

1. (4+4+4+4=16 points) Consider the points $A(1,0,0)$, $B(0,1,0)$, $C(0,0,2)$ and $D(-1,-1,0)$ in $\mathbb{R}^3$, given in Cartesian coordinates.

(a) Find an equation of the plane passing through A, B and C.

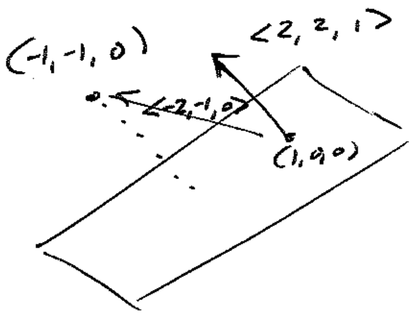


$$\langle 1,-1,0\rangle \times \langle 0,-1,2\rangle = \langle -2,-2,-1\rangle$$

$$-2x-2y-z=-2$$

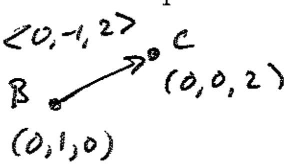
$$\boxed{2x+2y+z=2}$$

(b) Find the distance from the point D to the plane passing through A, B, C.



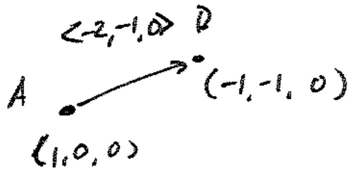
$$\left|\mathrm{Proj}_{\langle 2,2,1\rangle}\langle -2,-1,0\rangle\right| = \left|\frac{\langle -2,-1,0\rangle \cdot \langle 2,2,1\rangle}{|\langle 2,2,1\rangle|}\right| = \frac{|-4-2|}{\sqrt{4+4+1}} = \boxed{6/3}$$

(c) Find parametric equations for the lines BC and AD.



$$r_1(t) = \langle 0,1,0\rangle + \langle 0,-1,2\rangle t$$

$$\boxed{r_1(t) = \langle 0,\ 1-t,\ 2t\rangle}$$



$$r_2(t) = \langle 1,0,0\rangle + \langle -2,-1,0\rangle t$$

$$\boxed{r_2(t) = \langle 1-2t,\ -t,\ 0\rangle}$$

(d) Find the distance between the lines BC and AD.

plane through AD ∥ to BC has $\bar{n} = \langle -2,-1,0\rangle \times \langle 0,-1,2\rangle$
$= \langle -2,4,2\rangle$

$$\left|\mathrm{Proj}_{\bar{n}} \vec{AB}\right| = \left|\mathrm{Proj}_{\langle -1,2,1\rangle}\langle -1,1,0\rangle\right| = \frac{|\langle -1,1,0\rangle \cdot \langle -1,2,1\rangle|}{|\langle -1,2,1\rangle|} = \frac{1+2}{\sqrt{6}}$$

$$= \frac{3}{\sqrt{6}} = \boxed{\sqrt{3/2}}$$

2. (18 points) Match the following quadric equations with their graphs and give their name (e.g. *paraboloid*).

(i) $-x - y^2 + z^2 = 0$

$x = z^2 - y^2$ $\boxed{H}$

hyperbolic paraboloid

(ii) $x^2 - y^2 + z^2 = 0$

$y^2 = x^2 + z^2$ $\boxed{G}$

doublecone

(iii) $-x^2 + y^2 + z^2 = 1$

$x^2 = y^2 + z^2 - 1$ $\boxed{I}$

hyperboloid of 1 sheet

(iv) $x^2 - y^2 + z^2 = -1$ $\boxed{F}$

$y^2 = x^2 + z^2 + 1$

hyperboloid of two sheets

(v) $-x + y^2 - z^2 = 0$ $\boxed{C}$

$x = y^2 - z^2$

hyperbolic paraboloid

(vi) $x^2 + z^2 = 1$

cylinder $\boxed{B}$

(vii) $x^2 + 4y^2 + 4z^2 = 4$

ellipsoid $\boxed{D}$

(viii) $x^2 - y^2 + z^2 = 1$ $\boxed{A}$

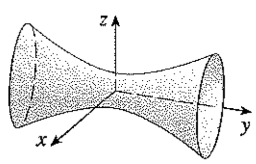
$y^2 = x^2 + z^2 - 1$

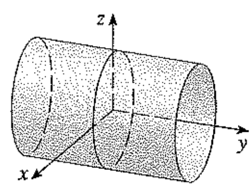
hyperbaloid of onesheet

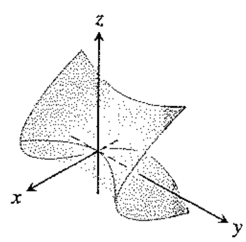
(ix) $x^2 - y + z^2 = 0$

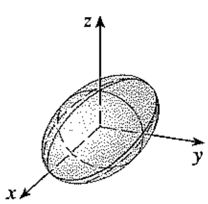
$y = x^2 + z^2$ $\boxed{E}$

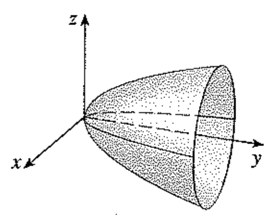
paraboloid

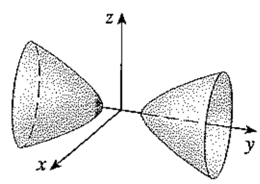
A 

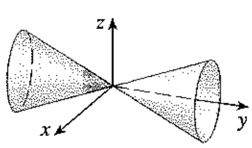
B 

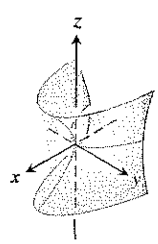
C 

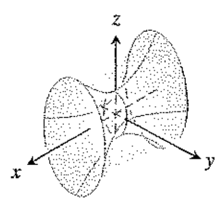
D 

E 

F 

G 

H 

I 

3. (5 points) Describe the surface $\rho^2 - 4\rho + 3 = 0$ in $\mathbb{R}^3$, where ρ is the radial coordinate in the spherical coordinate system.

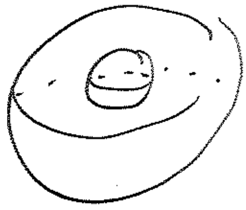
$$(\rho-3)(\rho-1)=0$$

$$\rho = 3, 1$$

Two spheres.

Sphere of radius 1.

Sphere of radius 3.

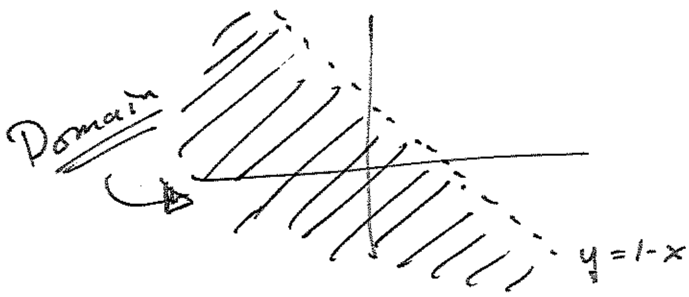


4. (5 points) Find and sketch the **domain** of the function

$$f(x,y) = \frac{x+y}{\sqrt{1-x-y}}$$

~~[illegible]~~ $1-x-y > 0$

$$y < 1-x$$



5. (5 points) Does the following limit exist? If so, what is its value?

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^2+y^2}$$

$\boxed{\text{Limit exists and is } 0}$

Polar coords: $\displaystyle \lim_{r\to 0} \frac{r^2\cos^2\theta \sin(r\sin\theta)}{r^2}$

$$\geq \lim_{r\to 0} \frac{r^2\cos^2\theta(-r)}{r^2}, \qquad \leq \lim_{r\to 0} \frac{r^2\cos^2\theta(r)}{r^2}$$

$$\lim_{r\to 0} -|r| = 0, \qquad \lim_{r\to 0} |r| = 0, \qquad \lim_{r\to 0} -|r| = 0, \qquad \lim_{r\to 0} |r| = 0$$

6. (5+5+5=15 points) Notice that the graph of any function $y = f(x)$ on the xy-plane can be parametrized as $\mathbf{r}(t) =< t, f(t) >$. Suppose that $f(x)$ has a continuous second derivative.

(a) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

$$r'(t) = \langle \frac{d}{dt} t, \frac{d}{dt} f(t) \rangle$$
$$= \langle 1, f'(t) \rangle$$
$$r''(t) = \langle \frac{d^2}{dt^2} t, \frac{d^2}{dt^2} f(t) \rangle$$
$$= \langle 0, f''(t) \rangle$$

(b) Find parametric equations of the tangent line to the curve at $(t_0, f(t_0))$ using the information in part (a).

Tangent line goes through $(t_0, f(t_0))$ in direction $\langle 1, f'(t_0) \rangle$

$$T(t) = \langle t_0, f(t_0) \rangle + \langle 1, f'(t_0) \rangle t$$
$$= \langle t_0 + t, f(t_0) + f'(t_0) t \rangle$$

$$\boxed{x(t) = t_0 + t \qquad y(t) = f(t_0) + f'(t_0) t}$$

(c) Show that the curvature is 0 at the points of inflection of this graph.

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|\langle 1, f'(t), 0 \rangle \times \langle 0, f''(t), 0 \rangle|}{|\langle 1, f'(t), 0 \rangle|^3}$$

$$= \frac{|\langle 0, 0, f''(t) \rangle|}{(1 + (f'(t))^2)^{3/2}}$$

$$= \frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}$$

If $f''(t) = 0$ then $K(t) = 0$.

7. (6+6+6=18 points) Let $f(x, y, z) = x^2 + y\ln(z+1)$, and $\mathbf{v} = \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$.

(a) Find $D_{\mathbf{v}} f(1, -1, 0)$.

$$\nabla f = \left\langle 2x,\ \ln(z+1),\ \frac{y}{z+1} \right\rangle$$

$$\nabla f(1,-1,0) = \left\langle 2,\ \ln(1),\ \frac{-1}{1} \right\rangle$$

$$= \langle 2,\ 0,\ -1 \rangle$$

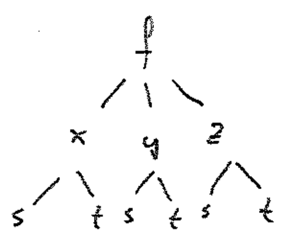
$$D_{\mathbf{v}} f = \left\langle \tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3} \right\rangle \cdot \langle 2, 0, -1 \rangle = \tfrac{2}{3} - \tfrac{2}{3} = 0$$

(b) Compute $\dfrac{\partial^2 f}{\partial z^2}$.

$$\frac{\partial f}{\partial z} = \frac{y}{z+1}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{-y}{(z+1)^2}$$

(c) If $x = st$, $y = s + t^2$, and $z = t - s^2$, find $\dfrac{\partial f}{\partial s}$ by using the Chain rule.

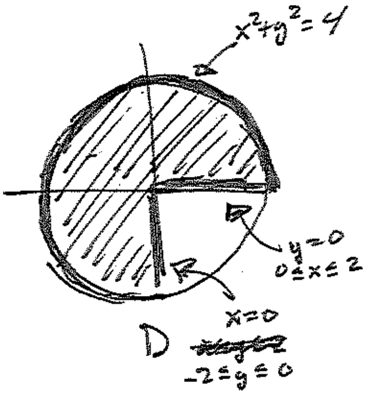


$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (2x)(t) + (\ln(z+1))(1) + \left(\frac{y}{z+1}\right)(-2s)$$

$$= \boxed{2st^2 + \ln(t-s^2+1) + \left(\frac{s+t^2}{t-s^2+1}\right)(-2s)}$$

8. (18 points) Let $f(x, y) = x^2 + 2x + y^2 + 2y$. Find the absolute maximum and minimum of f on the region D defined by the inequalities $0 \le r \le 2$, $0 \le \theta \le \frac{3\pi}{2}$ in polar coordinates.



Answer:
Global max $(\sqrt{2}, \sqrt{2})$ $f = 4+4\sqrt{2}$
Global min $(-1,-1)$ $f=-2$

Critical Points:

$0 = \nabla f = \langle 2x+2,\ 2y+2 \rangle$

$x=-1,\ y=-1$

$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$= 2 \cdot 2 - 0 > 0$

Note $(-1,-1)$ is in D

So $(-1,-1)$ is a max/min $\quad f(-1,-1) = -2$
$\frac{\partial^2 f}{\partial x^2} > 0$ so it is a min

Check boundary of D:

$x=0$: $f(0,y) = y^2+2y$

$0 = f'(0,y) = 2y+2$

$\boxed{-1 = y}$ ok

$f''(0,y) = 2$ so $(0,-1)$ is min on this part.

$y=0$: $f(x,0) = x^2+2x$

$0 = f'(x,0) = 2x+2$

$-1 = x$ ← not in region of interest. $(0 \le x \le 2)$

~~$f''(x,0) = 2$ so $(-1,0)$ is min~~

Check corners:
$f(0,0) = 0$
$f(0,-2) = 0$
$f(2,0) = 8$

$x^2+y^2=4$: Lagrange mult.

$\partial/\partial x$: $(\lambda 2x = 2x+2)y$

$\partial/\partial y$: $(\lambda 2y = 2y+2)x$

$x^2+y^2=4$ ← plug in

$\lambda 2xy = 2xy + 2y$
$\lambda 2xy = 2xy + 2x$

$2y = 2x$
$y = x$

$2x^2 = 4$

$x = \pm\sqrt{2} \Rightarrow y = \pm\sqrt{2}$

$f(\sqrt{2}, \sqrt{2}) = 4+4\sqrt{2}$
$f(-\sqrt{2}, -\sqrt{2}) = 4-4\sqrt{2}$
$f(-\sqrt{2}, \sqrt{2}) = 4$
$f(\sqrt{2}, -\sqrt{2}) = 4$

max is at $(\sqrt{2}, \sqrt{2})$ ~~and $(-\sqrt{2}, -\sqrt{2})$~~