KEY

	Calculus and Analytical Geometry I. Midterm	
Code : *Math 119* Acad.Year: *2010-2011* Semester : *Spring* Date : *26.3.2011* Time : *13:30* Duration : *120 minutes*	Last Name: Name : Department: Signature:	Student No: Section:
	8 QUESTIONS ON 8 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7	8		

Show your work! Please draw a $\boxed{\text{box}}$ **around your answers!**

1.(5 pts) Let $f(x) = \begin{cases} x\left(1 + x\cos\frac{1}{x}\right), & \text{if } x \neq 0 \\ c, & \text{if } x = 0. \end{cases}$

What value of c will make f continuous?

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \left(x + x^2\cos\frac{1}{x}\right) = 0$. Recall $-x^2 \leq x^2\cos\frac{1}{x} \leq x$ and squeezing thm

Hence $f(x)$ is continuous at $x = 0$ if $\boxed{c = 0}$

2. (5+5+5+5 pts) Evaluate the following limits, if they exist. Show your work. **Do not use L'Hospital's rule.**

(a) $\lim_{x\to\infty}\left(2x-\sqrt{4x^2-3x}\right)$.

$$\left\{ 2x-\sqrt{4x^2-3x} = \frac{(2x+\sqrt{4x^2-3x})(2x-\sqrt{4x^2-3x})}{2x+\sqrt{4x^2-3x}} \right.$$

$$= \frac{3x}{2x+2|x|\sqrt{1-\frac{3}{4x}}}$$

$$\lim_{x\to\infty}(2x-\sqrt{4x^2-3x}) = \lim_{x\to\infty}\frac{3x}{2x+2x\sqrt{1-\frac{3}{4x}}}, \quad |x| = x \text{ as } x\to\infty$$

$$\boxed{=\frac{3}{4}}$$

(b) $\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{1-x}$.

$$= \frac{\sqrt{1-1}}{1-0} = \frac{0}{1} = 0$$

(c) $\lim_{x\to 4}\frac{x-4}{2-\sqrt{x}}$.

$$= \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{2-\sqrt{x}} = -\lim_{x\to 4}(\sqrt{x}+2) = -4$$

(d) $\lim_{x\to 0}\frac{(\tan x)^2+\sin(x^3)}{2x^2}$.

$$= \lim_{x\to 0}\left[\frac{1}{2\cos^2 x}\left(\frac{\sin x}{x}\right)^2 + \frac{x}{2}\,\frac{\sin(x^3)}{x^3}\right]$$

$$= \frac{1}{2}\cdot 1^2 + 0\cdot 1$$

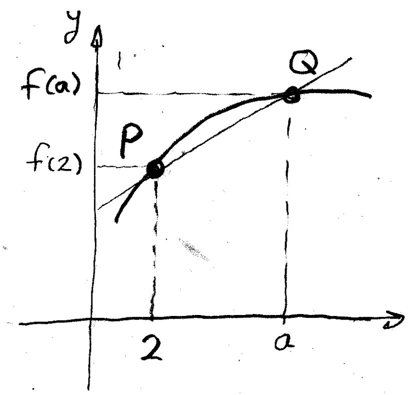
$$= \frac{1}{2} \qquad \text{since } \lim_{A\to 0}\frac{\sin A}{A} = 1$$

3. (5+5 pts) The following questions deal with definitions of the derivative.

(a) The function $y = f(x)$ has secant lines through the points $\big(2, f(2)\big)$ and $\big(a, f(a)\big)$ given by

$$y = \left(\frac{a^2 - 3a + 2}{a - 2}\right)(x - 2) + 5$$

for every a. What is the derivative $f'(2)$?



$$\boxed{f'(2) = \lim_{a \to 2} \frac{a^2 - 3a + 2}{a - 2} = 1}$$

$$m_P = f'(2) = \lim_{Q \to P} m_{\overline{PQ}} = \lim_{a \to 2} m_{\overline{PQ}}, \quad m_{\overline{PQ}} = \frac{a^2 - 3a + 2}{a - 2}$$

(b) If plugging into the definition of the derivative **immediately** yields

$$g'(1) = \lim_{h \to 0} \frac{\sqrt{h^2 + 2h + 4} - 2}{h}$$

then what is $g(x)$? ?

- ~~Choose $a = 0$ and $g(x) = \sqrt{x^2 + 2x + 4}$. Then~~

$$\cancel{g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt{h^2 + 2h + 4} - 2}{h}}$$

- Choose $a = 1$ and $g(x) = \sqrt{x^2 + 3}$. Then

$$g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0} \frac{\sqrt{(h+1)^2 + 3} - 2}{h}$$

4. (5+5+5+5 pts) Calculate the following derivatives.

(a) $\frac{d}{dx}((x^2+1)\sin x\cos x)$. $= \frac{d}{dx}\left[\frac{1}{2}(x^2+1)\sin 2x\right]$

$= x\sin 2x + \frac{1}{2}(x^2+1)\,2\cos 2x$

$= x\sin 2x + (x^2+1)\cos 2x$

or

$= 2x\sin x\cos x + (x^2+1)(\cos^2 x - \sin^2 x)$

(b) $\frac{d}{dx}\sec(\tan x + \sqrt{\sin x})$.

Let $u = \tan x + \sqrt{\sin x}$. Then $\frac{du}{dx} = \sec^2 x + \frac{1}{2}\frac{\cos x}{\sqrt{\sin x}}$ and

$\frac{d}{dx}\sec u = \frac{d}{du}(\sec u)\cdot\frac{du}{dx}$

$= \sec u\tan u \cdot \frac{du}{dx}$

(c) Find y' in terms of x and y if $2\cos x\sin y = 3$.

$-2\sin x\sin y + \cos x\cos y \cdot y' = 0$

$y' = \frac{\sin x}{\cos x}\frac{\sin y}{\cos y} = \tan x\tan y$

(d) Find y'' in terms of x and y if $2\cos x\sin y = 3$.

$y'' = \sec^2 x\tan y + \tan x\sec^2 y\cdot y'$

$y'' = \sec^2 x\tan y + \tan^2 x\tan y\sec^2 y$

5. (10 pts) Find the equation of the tangent line to the curve

$$y^2 + \sin(xy) - x^2 = \frac{\pi^2}{4}$$

at the point $(1, \frac{\pi}{2})$.

By implicit differentiation, we have

$$2yy' + (y + xy')\cos(xy) - 2x = 0$$

$$[2y + x\cos(xy)]y' - 2x + y\cos(xy) = 0$$

and

$$(\pi + \cos\frac{\pi}{2})y' = 2 - \frac{\pi}{2}\cos\frac{\pi}{2} \Rightarrow y' = \frac{2}{\pi} \text{ at } P(1, \frac{\pi}{2}).$$

Hence the required tangent line is

$$y - y_0 = m_P(x - x_0) \qquad \begin{cases} (x_0, y_0) = (1, \frac{\pi}{2}) \\ m_P = y'(1, \frac{\pi}{2}) = \frac{2}{\pi} \end{cases}$$

$$y - \frac{\pi}{2} = \frac{2}{\pi}(x - 1)$$

or

$$\boxed{y = \frac{\pi}{2} + \frac{2}{\pi}(x - 1)}$$

6. (10 pts) A ball is being filled with air at a rate of 3 $\frac{\text{cm}^3}{\text{s}}$. At what rate is the surface area changing when the ball has surface area 36 cm^2?

(Remember that $\text{SA} = 4\pi r^2$, and $\text{Vol} = \frac{4}{3}\pi r^3$.)

Let $SA := S(t) = 4\pi[r(t)]^2$. When $S = 36\ \text{cm}^2$ we have $36 = 4\pi r^2$ and $\boxed{r = \frac{3}{\sqrt{\pi}}}$.

On the other hand, $\text{Vol} := V(t) = \frac{4}{3}\pi[r(t)]^3$ and

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

When $\frac{dV}{dt} = 3\ \text{cm}^3/\text{s}$ and $r = \frac{3}{\sqrt{\pi}}$ we find that

$$3 = 4\pi \cdot \frac{9}{\pi} \frac{dr}{dt} \Rightarrow \boxed{\frac{dr}{dt} = \frac{1}{12}}$$

Therefore,

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \cdot \frac{3}{\sqrt{\pi}} \cdot \frac{1}{12} = 2\sqrt{\pi}$$

when $r = \frac{3}{\sqrt{\pi}}$ and $\frac{dr}{dt} = \frac{1}{12}$

7. (15 pts) Find the absolute minimum and absolute maximum of the function

$$f(x) = 4\sqrt{|x|} - x^2 + 1$$

on the interval $[-2, 2]$.

Endpoints $\boxed{f(-2) = f(2) = 4(\sqrt{2}-1)+1 \cong 2.66}$

f is differentiable for all $x \in (-2,2)$ except $x=0$.

So $x=0$ is a "critical point" at which the derivative d.n.e., where

$$\boxed{f(0) = 1}$$

For $x>0$: $f(x) = 4\sqrt{x} - x^2 + 1$, $f'(x) = \frac{2}{\sqrt{x}} - 2x$

For $x<0$: $f(x) = 4\sqrt{-x} - x^2 + 1$, $f'(x) = -\frac{2}{\sqrt{-x}} - 2x$

Therefore $f'(x) = 0$ at $x = \mp 1$ which are also critical points at which

$$f(-1) = f(1) = 4$$

Consequently, on $[-2, 2]$

- f achieves absolute maximum 4 at $x = \mp 1$
- f achieves absolute minimum 1 at $x = 0$

8. (10 pts) Use *linear approximation* to estimate the value of $(27.01)^{\frac{4}{3}}$.

Consider the tangent line to $f(x) = x^{4/3}$ at $a = 27$, i.e

$$L(x) = f(27) + f'(27)(x-27)$$

$$L(x) = 81 + 4(x-27) = 4x - 27$$

$$\begin{cases} f(27) = 27^{4/3} = 3^4 = 81 \\ f'(27) = \frac{4}{3}x^{1/3}\Big|_{x=27} = 4 \end{cases}$$

Hence

$$f(27.01) \simeq L(27.01) = 81.04$$