M E T U – N C C
Mathematics Group

	Calculus with Analytic Geometry Final Exam	
Code : MAT 120 Acad.Year : 2010-2011 Semester : Summer Instructors : A.D./E.G./O.K. Date : 15.08.2011 Time : 09.30 Duration : 150 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : 8 Questions on 8 Pages Total 100 Points	

1 (10)	2 (10)	3 (15)	4 (10)	5 (10)	6 (20)	7 (15)	8 (10)		

Q.1 (10 pts) Find the equation of the plane through the points $(1, 2, 0)$, $(0, -1, 1)$ and $(-1, 0, -1)$ in space.

Let $A = (1,2,0)$, $B = (0,-1,1)$, $C = (-1,0,-1)$

$\vec{AB} = (-1,-3,1)$, $\vec{AC} = (-2,-2,-1)$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & -3 & 1 \\ -2 & -2 & -1 \end{vmatrix} = 5i - 3j - 4k = (5,-3,-4)$$

Plane equation:

$\vec{n} \cdot (x-1, y-2, z-0) = 0$

$(5,-3,-4) \cdot (x-1, y-2, z) = 0$

$$\boxed{5(x-1) - 3(y-2) - 4z = 0}$$

Q.2 (10 pts) Find and classify all critical points of the function $f(x,y) = xye^{-x^2-y^2}$.

$$\nabla f = \left(ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2},\ xe^{-x^2-y^2} - 2y^2xe^{-x^2-y^2} \right)$$

$$= \left(y(1-2x^2),\ x(1-2y^2) \right) e^{-x^2-y^2}$$

$$\nabla f = \vec{0} \iff y(1-2x^2)=0 \quad \text{and} \quad x(1-2y^2)=0$$

$$\iff (y=0 \text{ and } x=0) \text{ or } \left(x = \mp\frac{1}{\sqrt{2}} \text{ and } y = \mp\frac{1}{\sqrt{2}} \right)$$

1 pt: $(0,0)$

4 points: $\left(\frac{-1}{\sqrt2}, \frac{-1}{\sqrt2}\right)$, $\left(\frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)$, $\left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}\right)$ and $\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$$f_{xx} = -2xye^{-x^2-y^2} - 4xye^{-x^2-y^2} + 4x^3ye^{-x^2-y^2} = (4x^3-6x)y\,e^{-x^2-y^2}$$

$$f_{yy} = (4y^3-6y)xe^{-x^2-y^2}$$

$$f_{xy} = e^{-x^2-y^2} - 2y^2e^{-x^2-y^2} - 2x^2e^{-x^2-y^2} + 4x^2y^2e^{-x^2-y^2}$$

$$= (1-2x^2-2y^2+4x^2y^2)e^{-x^2-y^2}$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = \left(e^{-x^2-y^2}\right)^2 \Big[(4x^3-6x)y \cdot (4y^3-6y)x - (1-2x^2-2y^2+4x^2y^2)^2 \Big]$$

$D(0,0) = -1 < 0 \Rightarrow \boxed{(0,0) \text{ is a saddle}}$

D has only even powers. $D\left(\mp\frac{1}{\sqrt2}, \mp\frac{1}{\sqrt2}\right) = e^{-2}\Big[\left(\frac{2}{\sqrt2} - \frac{6}{\sqrt2}\right)\frac{1}{\sqrt2}\left(\frac{2}{\sqrt2} - \frac{6}{\sqrt2}\right)\frac{1}{\sqrt2} - (\underbrace{1-1-1+1}_{0})^2 \Big] > 0$

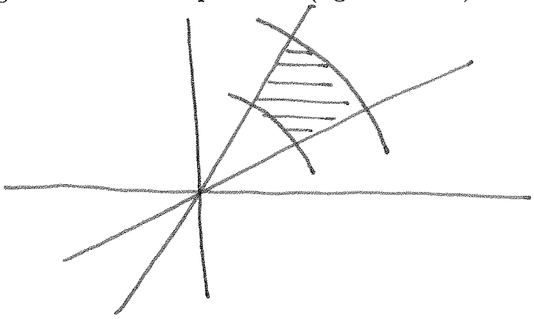
$f_{xx}\left(\frac{-1}{\sqrt2}, \frac{-1}{\sqrt2}\right) = \left(\frac{-2}{\sqrt2} + \frac{6}{\sqrt2}\right)\left(\frac{-1}{\sqrt2}\right)e^{-1} < 0 \Rightarrow \boxed{\text{local max at } \left(\frac{-1}{\sqrt2}, \frac{-1}{\sqrt2}\right)}$

$f_{xx}\left(\frac{-1}{\sqrt2}, \frac{+1}{\sqrt2}\right) = \left(\frac{-2}{\sqrt2} + \frac{6}{\sqrt2}\right)\left(\frac{1}{\sqrt2}\right)e^{-1} > 0 \Rightarrow \boxed{\text{local min at } \left(\frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)}$

$f_{xx}\left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}\right) = \left(\frac{2}{\sqrt2} - \frac{6}{\sqrt2}\right)\left(\frac{-1}{\sqrt2}\right)e^{-1} > 0 \Rightarrow \boxed{\text{local min at } \left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}\right)}$

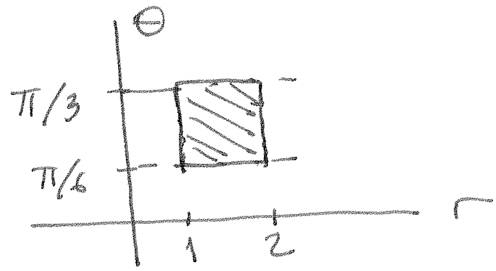
$f_{xx}\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right) = \left(\frac{2}{\sqrt2} - \frac{6}{\sqrt2}\right)\left(\frac{1}{\sqrt2}\right)e^{-1} < 0 \Rightarrow \boxed{\text{local max at } \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)}$

Q.3 (7+8=15 pts) Let $D = \left\{(x,y) : 1 \le x^2 + y^2 \le 4, \frac{1}{\sqrt{3}} \le \frac{y}{x} \le \sqrt{3}\right\}$ be the region in the first quadrant (figure below)



(a) Using the polar coordinates, transform the region D into the r-θ domain, and sketch the obtained region.

$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ and $1 \leq r \leq 2$



(b) Evaluate the following integral $\iint_D 3\sqrt{x^2+y^2}dA$ based on the transformation proposed in (**a**).

$$\iint_D 3\sqrt{x^2+y^2}\,dA = \int_{\pi/6}^{\pi/3}\int_1^2 3r \cdot r\,dr\,d\theta$$

$$= \left(\int_{\pi/6}^{\pi/3} d\theta\right)\left(r^3\Big|_1^2\right)$$

$$= \left(\frac{\pi}{3} - \frac{\pi}{6}\right)(8-1)$$

$$= \boxed{7\pi/6}$$

Q.4 (10 pts) For each of the following vector fields, check whether it is conservative or not. If it is, find a potential function.

(a) $\mathbf{F}(x,y) = 2x^2\cos(xy)\,\mathbf{i} + x^3\cos(xy)\mathbf{j}$.

$= P\vec{i} + Q\vec{j}$

$P_y = -x \cdot 2x^2 \sin(xy)$

$Q_x = 3x^2\cos(xy) - yx^3\sin(xy)$

$P_y \neq Q_x \Rightarrow \boxed{\text{not conservative}}$

(b) $\mathbf{F}(x,y,x) = (y^2 + yze^x)\,\mathbf{i} + (2xy + ze^x)\,\mathbf{j} + ye^x\mathbf{k}$.

$= P\vec{i} + Q\vec{j} + R\vec{k}$

$P_y = 2y + ze^x = Q_x$

$P_z = ye^x = R_x$

$Q_z = e^x = R_y$

test holds everywhere on a simply connected domain ($\mathbb{R}^3$) $\Rightarrow$ field is conservative.

Suppose $\vec{F} = \nabla\phi(x,y,z)$

$\phi_z = ye^x \Rightarrow \phi = yze^x + f(x,y)$

$\phi_x = yze^x + f_x = y^2 + yze^x$

$\Rightarrow f_x = y^2 \Rightarrow f(x,y) = xy^2 + h(y) \Rightarrow \phi = yze^x + xy^2 + h(y)$

$\phi_y = ze^x + 2xy + h'(y) = 2xy + ze^x$

$\Rightarrow$ can take $h = 0$

$$\boxed{\phi = yze^x + xy^2}$$

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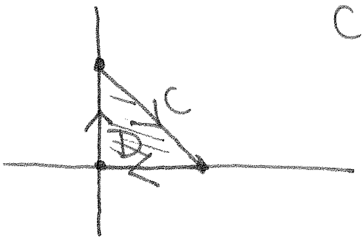
Q.5 (10 pts) Find the work $W = \oint_C \left(e^{\sin(x)} - yx\right) dx + \left(x^2 y + \frac{1}{e^{\sqrt{y}}}\right) dy$ done on a particle along the closed curve C, which starts at $(0,0)$, moves along y-axis to $(0,1)$, then along the segment line to $(1,0)$, and then along the x-axis to the starting point (*Hint: Use Green's theorem*).



C is oriented clockwise

$$\oint_C P\,dx + Q\,dy = -\iint_D (Q_x - P_y)\,dx\,dy$$

$$= -\iint_D 2xy + x \; dx\,dy$$

$$= -\int_0^1 \left(\int_0^{1-x} 2xy + x \; dy \right) dx$$

$$= -\int_0^1 \left(xy^2 + xy \Big|_0^{1-x} \right) dx$$

$$= -\int_0^1 x(1-x)^2 + x(1-x)\; dx$$

$$= -\int_0^1 x^3 - 3x^2 + 2x \; dx = -\frac{1}{4} + 1 - 1 = \boxed{\frac{-1}{4}}$$

Q.6 (3+3+3+3+4+4=20 pts) Test for convergence or divergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2(n)}$

$$0 \leq \cos^2(n) \leq 1$$

$$n + n\cos^2 n \leq 2n$$

$$\Rightarrow \frac{1}{n + n\cos^2 n} \geq \frac{1}{2n}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (by p-test), so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ also diverges.

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$ ~~Apply~~ the root test.

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left(\frac{2n}{n+1}\right)^5 = 32 > 1 \Rightarrow \text{series diverges}$$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n}$ This is an alternating series.

Let $f(x)=\frac{\ln x}{x}$. Then, $f'(x)=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}$

Since $\ln x > 1$ for $x>e$, we can say that $b_n=f(n)$ is decreasing for $n\geq 3$

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty}\frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty}\frac{1/x}{1}=0.$$

Both conditions of the alternating series test are satisfied.

$\Rightarrow$ series converges.

(d) $\displaystyle\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right)$

Use limit comparison with $1/n^2$. Notice that $\ln(1+\frac{1}{n^2})>0$.

$$\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n^2})}{1/n^2}=\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x^2})}{1/x^2}=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x^2}}\cdot\cancel{\frac{-2}{x^3}}}{\cancel{\frac{-2}{x^3}}}=1\neq 0,\infty.$$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-test $\Rightarrow \sum_{n=1}^{\infty}\ln(1+\frac{1}{n^2})$ converges

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$

Use limit comparison with $1/n$ (Series is positive)

$$\lim_{n\to\infty}\frac{1/n^{1+1/n}}{1/n}=\lim_{n\to\infty}\frac{1}{n^{1/n}}=\lim_{x\to\infty}\frac{1}{x^{1/x}}$$

$\ln(x^{1/x})=\frac{1}{x}\ln x$ and $\lim_{x\to\infty}\frac{1}{x}\ln x=0 \Rightarrow \lim_{x\to\infty}x^{1/x}=e^0=1 \Rightarrow \lim_{x\to\infty}\frac{1}{x^{1/x}}=1 \neq 0,\infty$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$ diverges.

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

Use limit comparison with $1/n^2$. (This is a positive series)

$$\lim_{n\to\infty}\frac{\sqrt{n^2-1}/(n^3+2n^2+5)}{1/n^2}=\lim_{x\to\infty}\frac{\sqrt{x^2-1}/(x^3+2x^2+5)}{1/x^2}$$

$$=\lim_{x\to\infty}\frac{x\sqrt{1-1/x^2}/x^3(1+\frac{2}{x}+\frac{5}{x^3})}{1/x^2}=1\neq 0,\infty$$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-test so $\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges

Q.7 (5+10=15 pts) Let $f(x) = \dfrac{x^2 - x + 5}{(3x+1)(x-2)^2}$ be a rational function

(a) Find the partial fraction expansion (from MAT 119) of the function $f(x)$.

$$\frac{x^2-x+5}{(3x+1)(x-2)^2} = \frac{A}{3x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Then, $x^2 - x + 5 = A(x-2)^2 + B(3x+1)(x-2) + C(3x+1)$

put $x=2 \Rightarrow 4-2+5 = 7C \Rightarrow \boxed{C=1}$

put $x = -1/3 \Rightarrow \frac{1}{9} + \frac{1}{3} + 5 = \frac{49A}{9} \Rightarrow \boxed{A=1}$

Take a derivative:

$2x - 1 = 2A(x-2) + 3B(x-2) + B(3x+1) + 3C$

put $x=2$ $\quad 3 = 7B + 3 \Rightarrow \boxed{B=0}$

(b) Based on the partial fraction expansion, find the power series expansion of the function $f(x)$ around the point $a = 0$. State the interval of convergence.

$$\cancel{\frac{1}{3x+1} = \frac{1}{3\left(x+\frac{1}{3}\right)} = \frac{1}{3}\Bigg(}$$

$$\frac{1}{1+3x} = 1 - 3x + (3x)^2 - (3x)^3 + (3x)^4 - \cdots$$

for $|3x| < 1 \Leftrightarrow |x| < \frac{1}{3}$.

$$\frac{1}{(x-2)^2} = -\left(\frac{1}{x-2}\right)'$$

$$\frac{1}{x-2} = \frac{-1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = -\frac{1}{2}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\right)$$

for $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$.

$$\frac{1}{(x-2)^2} = \frac{+1}{2}\left(\frac{1}{2} + 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 + \cdots\right)$$

So

$$\frac{x^2-x+5}{(3x+1)(x-2)^2} = \frac{1}{1+3x} + \frac{1}{(x-2)^2} = \left(\left(1+\frac{1}{4}\right) + \left(-3 + \frac{2}{2^2}\right)x + \left((-3)^2 + \frac{3}{2^3}\right)x^2 + \cdots + \left((-3)^n + \frac{(n+1)}{2^{n+1}}\right)x^n + \cdots\right)$$

for $x \in (-1/3, 1/3)$.

Q.8 (10 pts) Estimate the integral $\int_0^{0.1} \left(e^{x^3} + f(x)\right) dx$ by using a power series expansion up to the x^4 term, where the function $f(x)$ is supposed to be a smooth function with the properties $f(0) = 5$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = 3$ and $f^{(4)}(0) = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

So $e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \cdots$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}}{4!}x^4 + \cdots$$

$$= 5 + x + \frac{3}{3!}x^3 + \cdots$$

$$= 5 + x + \frac{x^3}{2} + \cdots$$

$$\int_0^{0.1} \left(e^{x^3} + f(x)\right) dx \simeq \int_0^{0.1} 1 + x^3 + 5 + x + \frac{x^3}{2} \, dx$$

$$= \int_0^{0.1} 6 + x + \frac{3}{2}x^3 \, dx$$

$$= 6x + \frac{x^2}{2} + \frac{3}{8}x^4 \Big|_0^{0.1}$$

$$= 0.6 + \frac{0.01}{2} + \frac{3}{8}(0.0001)$$

$$= 0.6 + 0.005 + 0.0000375$$

$$\boxed{= 0.605375}$$